

# Affine Weyl group alcoves and the geometry of the unitary dual

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# Outline

Alcoves and  $\widehat{G}_U$

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Introduction

Integer parts...

Integer parts...

...and also ordering

...and ordering

Partitioning  $\mathbb{R}^n$  into faces

Faces

Faces and unitary representations

Group repns

Enumerating faces and unitary

FPP faces

Slides eventually at

<http://www-math.mit.edu/~dav/paper.html>

# What's this about?

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I'll talk about decomposing  $\mathbb{R}^n$  using **symmetries**.

Question is **how can you use symmetries to put any vector into the simplest possible form?**

Simple version: symms are **sign changes** of some coords, and **adding an integer** to a coord.

Next: add the symms **exchanging** any two coords.

Having tried to explain the **simplification** process in those two examples, I will talk about seeing the **affine Weyl group** as a generalization.

I'll finish with **mathematical reasons** for looking at such simplification problems.

# Reducing modulo $\mathbb{Z}$

How can you **simplify**  $v \in \mathbb{R}^n$  by **adding ints** to any coord and **chging sgns** of any coord?

First process allows moving any  $v$  to

$$W\overline{\mathcal{A}} =_{\text{def}} [-1/2, 1/2]^n.$$

$$(8/3, -4/5, 2) \xrightarrow{+(-3, 1, -2)} (-1/3, 1/5, 0).$$

Then the second process allows moving  $v$  to

$$\overline{\mathcal{A}} =_{\text{def}} [0, 1/2]^n.$$

$$(-1/3, 1/5, 0) \xrightarrow{(-, +, \pm)} (1/3, 1/5, 0).$$

# Let's write that as a Theorem

Two symmetry groups of  $\mathbb{R}^n$ :

Define  $T = \mathbb{Z}^n$ , translations of  $\mathbb{R}^n$  by integers.

Define  $W = (\pm 1)^n$ , coord sign changes in  $\mathbb{R}^n$ .

Recall  $\overline{\mathcal{A}} = [0, 1/2]^n$ ,  $W\overline{\mathcal{A}} = [-1/2, 1/2]^n$ .

## Theorem

1. For all  $v \in \mathbb{R}^n$   $\exists t \in \mathbb{Z}^n$  so  $v^1 =_{\text{def}} t + v \in W\overline{\mathcal{A}}$ .
2.  $t$  is **unique** except in coords with  $v_i \in \mathbb{Z} + 1/2$ .
3. For all  $v^1 \in W\overline{\mathcal{A}}$   $\exists w \in \pm 1^n$  so  $w \cdot v^1 =_{\text{def}} v^0 \in \overline{\mathcal{A}}$ .
4.  $w$  is **unique** except in coords where  $v_i^1 = 0$ .
5.  $v_0$  is **unique**.

Symm grp we want is  $W \ltimes T$ , a **semidirect product**.

This is an **affine Weyl group** of type  $(\widetilde{A}_1)^n$ .

But the main point is statements in **Theorem**.

# Let's draw that as a picture

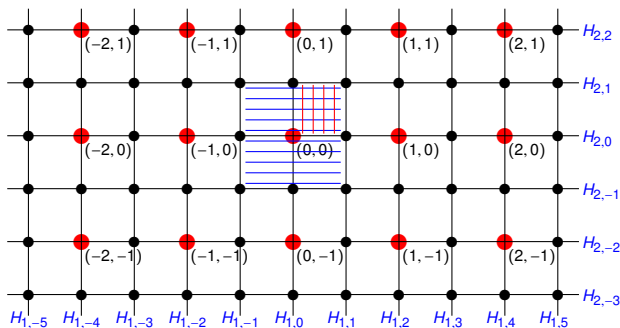
I'm interested in the **hyperplanes** in  $\mathbb{R}^n$

$$H_{i,m} = \{v \in \mathbb{R}^n \mid 2v_i = m\} \quad (1 \leq i \leq n, \quad m \in \mathbb{Z}).$$

For each hyperplane, I'm interested in the **reflection**

$$\begin{aligned} s_{i,m}(v) &= v - (2v_i - m)e_i \\ &= (v_1, \dots, v_{i-1}, -v_i + m, v_{i+1}, \dots, v_n). \end{aligned}$$

$s_{i,m}$  **chgs sign** of  $i$ th coord and **translates** by  $m$ .



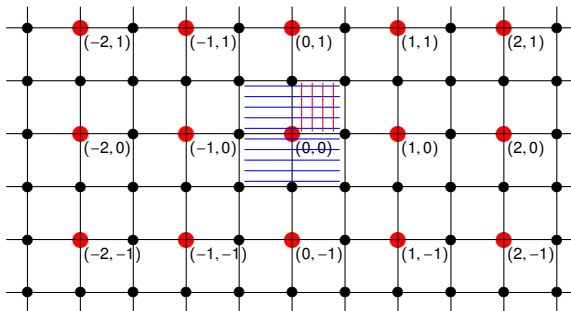
# How does the picture prove the theorem?

Start with **any**  $v \in \mathbb{R}^n$ .

Want to use **hyperplane reflections** to move  $v$  to

$$\overline{\mathcal{A}} = \text{[Diagram of a unit square with a red dot at the bottom-left corner and a red grid inside.]}$$

Whenever  $v$  is on the wrong side of a hyperplane  $H_{i,m}$  from  $\overline{\mathcal{A}}$ , **reflect  $v$  in that hyperplane**, moving it closer to  $\mathcal{A}$ .



# That wasn't complicated enough to be *math*

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Add **more** symmetries: **interchanging** coords of  $v$ .

How can you **simplify**  $v \in \mathbb{R}^n$  by **adding integers**, **chging coord sgns**, and **permuting** coords?

**First process** (still) allows moving any  $v$  to

$$W\overline{\mathcal{A}} =_{\text{def}} [-1/2, 1/2]^n, \quad \text{vol}(W\overline{\mathcal{A}}) = 1.$$

$$(7/4, -3/5, 3/2) \xrightarrow{+(-2, 1, -2)} (-1/4, 2/5, -1/2).$$

**Second processes** moves  $v$  to (smaller) cube

$$\overline{\mathcal{C}} =_{\text{def}} [0, 1/2]^n, \quad \text{vol}(\overline{\mathcal{C}}) = 1/2^n$$

$$(-1/4, 2/5, 1/2) \xrightarrow{+(-2, 1, -2)} (1/4, 2/5, 1/2).$$

**Third processes** move  $v$  to (smallest)

$$\overline{\mathcal{A}} =_{\text{def}} \{v \in \mathbb{R}^n \mid 1/2 \geq v_1 \geq v_2 \geq \cdots \geq v_n \geq 0\}.$$

$$(-1/4, 2/5, -1/2) \longrightarrow (1/2, 2/5, 1/4).$$

$\overline{\mathcal{A}}$  is an  **$n$ -simplex**, volume =  $1/(2^n \cdot n!)$

# This too is a Theorem

Define  $T = \mathbb{Z}^n$ , translations of  $\mathbb{R}^n$  by integers.

Define  $W = S_n \ltimes (\pm 1)^n$ , coord perms and sign changes.

Our new  $\overline{\mathcal{A}} = \{1/2 \geq v_1 \geq \cdots \geq v_n \geq 0\}$ ,

$W\overline{\mathcal{A}} = [-1/2, 1/2]^n$ .

Unit cube  $W\overline{\mathcal{A}}$  is union of  $2^n \cdot n!$  translates of simplex  $\overline{\mathcal{A}}$ .

## Theorem

1. For all  $v \in \mathbb{R}^n$   $\exists t \in \mathbb{Z}^n$  so  $v^1 =_{\text{def}} t + v \in W\overline{\mathcal{A}}$ .
2.  $t$  is **unique** except in coords with  $v_i \in \mathbb{Z} + 1/2$ .
3. For all  $v^1 \in W\overline{\mathcal{A}}$   $\exists w \in W$  so  $w \cdot v^1 =_{\text{def}} v^0 \in \overline{\mathcal{A}}$ .
4.  $w$  is **unique** unless  $v_i^1 = 0$  or  $\pm v_i^1 \pm v_j^1 = 0$ .
5.  $v_0$  is **unique**.

Symmetry grp we want is  $W \ltimes T$ , a **semidirect product**.

This is an **affine Weyl group** of type  $\widetilde{B}_n$ .

But the main point is statements in **Theorem**.

# Draw the new Theorem as a picture

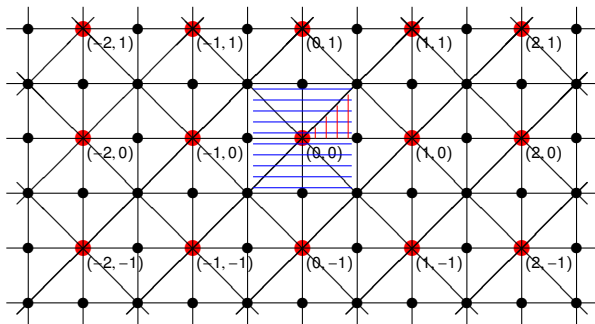
New **hyperplanes** are

$$H_{i,\pm j,m} = \{v \in \mathbb{R}^n \mid v_i \pm v_j = m\} \quad (1 \leq i, j \leq n, \quad m \in \mathbb{Z}).$$

For each hyperplane, we want the **reflection**

$$s_{i,\pm j,m}(\cdots, v_i, \cdots, v_j, \cdots) = (\cdots, \pm v_j + m, \cdots, \pm v_i \pm m, \cdots).$$

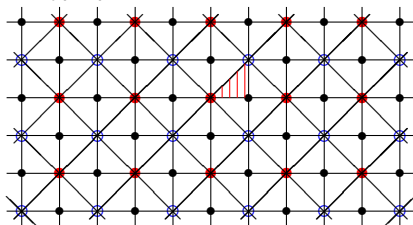
$s_{i,\pm j,m}$  **interchanges**  $i$ th and  $j$ th coords, multiplying both by  $\pm$ , and **translates** by  $m(e_i \pm e_j)$ .



# What's a face?

The **reflection hyperplanes** (like  $\{v_i + v_j = m\}$ ) each divide  $\mathbb{R}^n$  into **three pieces**: the hyperplane itself, and two open pieces.

These hyperplanes divide  $\mathbb{R}^n$  into **faces**. Here's  $\mathbb{R}^2$ .



Each open triangle is a face, called an **alcove**. An alcove has three kinds of 1-diml faces as edges, and three kinds of 0-diml faces as vertices.

There are **three kinds** of 0-diml faces:

1. **integral** ( $p, q$ ) ( $p$  and  $q$  in  $\mathbb{Z}$ );
2. **half-integral** ( $p + 1/2, q + 1/2$ ); and
3. **mixed** ( $p + 1/2, q$ ) or ( $p, q + 1/2$ ).

There are **three kinds** of 1-diml faces (black open intervals):

1. horiz or vert, with one red and one black endpoint;
2. horiz or vert, with one blue and one black endpoint; and
3. diagonal (always with one red and one blue endpoint).

# Everything you always wanted to know about faces

$T = \mathbb{Z}^n$ , transl of  $\mathbb{R}^n$ ;  $W = S_n \ltimes (\pm 1)^n = \text{type } B_n \text{ Weyl group.}$

$\widetilde{W} = W \ltimes T = \text{affine Weyl group.}$

Everything below works for  $W = \text{any Weyl group, } T = \text{root lattice.}$

An **alcove** is a conn component of  $\mathbb{R}^n$  – (all refl hyperplanes).

**Theorem**  $\widetilde{W}$  acts **simply transitively** on alcoves.

1. The **fundamental alcove**  $\mathcal{A}$  is the  $n$ -simplex

$$\mathcal{A} = \{1/2 \geq v_1 \geq \cdots \geq v_n \geq 0\}.$$

2. The  $n + 1$  vertices of  $\mathcal{A}$  are  $f_m = (\underbrace{1/2, \dots, 1/2}_{m \text{ terms}}, \underbrace{0, \dots, 0}_{n - m \text{ terms}}).$

3. Each alcove is an  $n$ -simplex, so has  $\binom{n+1}{d+1}$   $d$ -faces.

4. Every  $d$ -diml face is a  $d$ -face of some alcove.

This theorem provides a **computer-effective way** to list all faces.

I'll return to that after explaining why one might want a list of faces.

# And now for something completely different

Alcoves and  $\widehat{G}_U$

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My favorite problem in the whole world is the **unitary dual problem**.

Start with a group  $G$ ; look for **all ways that  $G$  can act by isometries of Hilbert spaces**.

**Quantum mech systems** live on Hilbert space, so **unitary rep  $\leftrightarrow$  symmetry of quantum systems**.

**How can you look for unitary reps?**

I'll explain how looking for unitary reps of **simple Lie groups** leads to geometry of faces.

# Two important subgroups for $GL(n, \mathbb{R})$

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Pure mathematicians like **eigenvalues** of matrices.

Google, our applied colleagues  $\rightsquigarrow$  this is **misguided**.

$K(\mathbb{R}) = O(n)$  = orthogonal group,

$A$  = **positive** diagonal matrices,

$A^+$  = positive diag mats with **decreasing** entries.

Any **invertible**  $n \times n$  real  $g$  has a **polar decomposition**

$$g = k_1 a k_2, \quad (a \in A^+, \quad k_i \in O(n)).$$

Matrix  $a$  is **unique**. Diagonal entries of  $a$  are the **singular values** of  $g$ . Largest singular value is

$$a_1 = \max_{v \in \mathbb{R}^n \setminus 0} (\|gv\|/\|v\|),$$

the **largest amount** that  $g$  can **stretch** a vector.

Similarly,  $a_n$  is the **least** that  $g$  can **shrink** a vector.

Since  $K(\mathbb{R})$  is **compact**, polar decomp says that  $A$  **enumerates all the ways to go to infinity in  $GL(n, \mathbb{R})$** .

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# So what can you do with $KA^+K$ ?

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$K = O(n)$  = orthogonal group,

$A^+$  = positive diag mats with decreasing entries.

Study harmonic functions on the unit disc by boundary values:  
limiting behavior in radial directions.

Same applies to functions on  $GL(n, \mathbb{R}) = KA^+K$ : helps to study  
limiting behavior in the  $A^+$  variable.

(approximate) Theorem (Harish-Chandra). Nice fn  $\phi$  on  $GL(n, \mathbb{R})$   
is exponential at infinity: have an asymptotic expansion

$$\phi(k_1 a k_2) \sim c(k_1, k_2) a^\nu + \text{lower terms}, \quad (a \in A^+ \rightarrow \infty)$$

with  $\nu \in \mathbb{C}^n$ . Here  $a^\nu = a_1^{\nu_1} \cdots a_n^{\nu_n}$ .

HC/Langlands idea: reps of  $GL(n, \mathbb{R})$  are indexed by  $\nu \in \mathbb{C}^n$   
describing their asymptotic behavior at infinity.

which reps are unitary  $\leftrightarrow$  which face  $\nu$  is in!

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# How do you make reps of $GL(n, \mathbb{R})$ ?

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Reps of  $G$  on fns on homogeneous spaces  $G/H$ .

Better: sections of vector bundles  $\mathcal{E} \rightarrow G/H$ .

Best space to use for  $GL(n, \mathbb{R})$ :

$X =$  complete flags  $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{R}^n$ .

$X$  has  $n$  real line bundles  $\mathcal{E}_i$ , fiber  $V_i/V_{i-1}$ .

$\nu \in \mathbb{R}^n \rightsquigarrow$  real line bundle  $\mathcal{E}_\nu = |\mathcal{E}_1|^{\nu_1} \otimes \cdots \otimes |\mathcal{E}_n|^{\nu_n}$

$\pi_\nu =$  sections of  $\mathcal{E}_\nu \otimes D^{1/2}$ , nice rep of  $GL(n, \mathbb{R})$ .

Here  $D^{1/2}$  is half-density bundle on  $X$ , useful normalization.

If  $\rho = ((n-1)/2, (n-3)/2, \dots, -(n-1)/2)$ , then  $D^{1/2} = \mathcal{E}_\rho$ .

**Theorem** (HC, Helgason, Helgason-Johnson). Say  $\nu_1 \geq \cdots \geq \nu_n$ .

1.  $\pi_\nu$  has asymptotic behavior  $a^{\nu-\rho}$  at infinity on  $A^+$ .
2.  $\pi_\nu$  bdd  $\iff \nu \in \rho - (\text{nonneg combs of pos roots } e_i - e_j)$ .
3.  $\pi_\nu$  herm  $\iff \nu = (\nu_1, \dots, \nu_m, \{0\}, -\nu_m, \dots, -\nu_1)$ .
4. In (3), whether  $\pi_\nu$  is unitary  $\iff$  which face  $\nu$  belongs to.

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# How to classify unitary reps of $GL(2m, \mathbb{R})$

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Unitary reps of  $GL(2m, \mathbb{R})$  indexed by **some faces** in

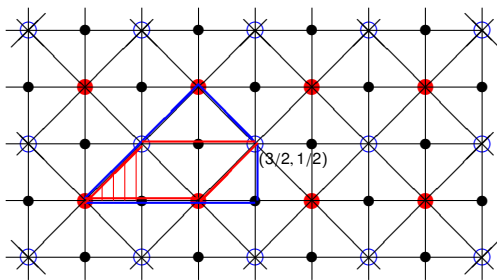
$$v = (v_1, \dots, v_m), \quad v_1 \geq \dots \geq v_m \geq 0$$

$$v_1 + v_2 + \dots + v_p \leq (m-1/2) + (m-3/2) + \dots + (m-(2p-1)/2)$$

So to describe unitary representations, need to

1. **enumerate finite #** faces satisfying inequalities; and
2. for each face, **test** whether **one**  $v$  in face is unitary.

Test (2) is possible using **atlas** software.



**Blue quadrilateral** is the candidates allowed by Helgason-Johnson: 7 alcoves, 29 faces. **Red parallelogram FPP** is a better bound found by Dan Barbasch: 4 alcoves, 19 faces.

Same ideas  $\rightsquigarrow$  unitary duals for **all** real reductive  $G$ .

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# So what's the program?

General  $G$ : **pos roots**  $R^+ \subset \mathbb{R}$  **vec space**  $\mathfrak{h}_{\mathbb{R}}^*$  replaces  $\mathbb{R}^n$ .

**Weyl group**  $W(R)$  replaces  $S_n \ltimes \{\pm 1\}^n$ .

**Root lattice**  $\mathbb{Z}R$  replaces  $\mathbb{Z}^n$ .

**Hyperplanes** are  $H_{\alpha^\vee, m} = \{\gamma \in \mathfrak{h}^* \mid \langle \gamma, \alpha^\vee \rangle = m\}$ .

**FPP** is  $\{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \gamma, \alpha^\vee \rangle \in [0, 1], \text{ all } \alpha \text{ simple}\}$ .

Need to

1. **compute** partition of FPP into faces
2. for one  $v$  in each face, **test** unitarity of **finitely many** reps of  $\text{infl char } v$ .

For  $E_7$ , number of faces in FPP is about **38 million**;  
**compute** them in few hours.

For  $E_8$ , number of faces in FPP is about **30 billion**;  
**compute** in a month or so.

**test** is harder...

# Extended affine Weyl group

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$R$  = root system in  $\mathfrak{h}_{\mathbb{R}}^*$   $\rightsquigarrow$   $W(R)$  = Weyl group.

$\mathbb{Z}R$  = lattice  $\rightsquigarrow$   $\widetilde{W}(R) = W(R) \ltimes \mathbb{Z}R$  **affine Weyl group**.

$\Xi$  = fundamental weights in  $\mathfrak{h}_{\mathbb{R}}^*$

$\mathbb{Z}\Xi \rightsquigarrow \widetilde{W}(R)^{\text{ext}} = W(R) \ltimes \mathbb{Z}\Xi$  **extended aff Weyl grp**.

Like  $\widetilde{W}(R)$ ,  $\widetilde{W}(R)^{\text{ext}}$  **permutes hyperplanes**  $H_{\alpha^\vee, m}$ .

$$\widetilde{W}(R)^{\text{ext}} / \widetilde{W}(R) \simeq \mathbb{Z}\Xi / \mathbb{Z}R \simeq \widehat{Z(G_{sc})}.$$

(elt of  $\mathbb{Z}\Xi$ )  $\rightsquigarrow$  (fin diml  $G_{sc}$  rep)  $\rightsquigarrow$  (char of  $Z(G_{sc})$ ).

(elt of  $\mathbb{Z}R$ )  $\rightsquigarrow$  (fin diml  $G_{ad}$  rep)  $\rightsquigarrow$  (triv char of  $Z(G_{sc})$ ).

**Proposition**. For any  $\nu \in \mathfrak{h}_{\mathbb{R}}^*$  there is a  $\tau \in \mathbb{Z}\Xi$  with  $\nu + \tau$  in the **FPP**. If  $\nu$  is **regular**, then  $\tau$  is **unique**.

**Proof**. The FPP  $\{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \gamma(\alpha_i^\vee) \in [0, 1] \text{ all } \alpha_i^\vee \text{ simple}\}$  is the **unit cube** in fundamental weight coordinates.

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# Alcoves in the FPP

Unitary dual of  $G(\mathbb{R})$  is built from a finite number of pieces, each a closed union of FPP faces

So need to understand those faces.

Recall that the (open) fundamental alcove is

$$\mathcal{A} = \{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \gamma(\alpha^\vee) + m > 0 \quad (\alpha^\vee > 0 \text{ coroot}, m \geq 0 \text{ integer}) \\ -\gamma(\beta^\vee) + n > 0 \quad (\beta^\vee > 0 \text{ coroot}, n > 0 \text{ integer})\}$$

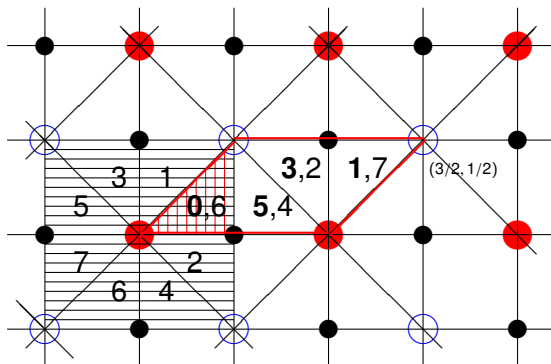
Recall also that the closed fundamental alcove  $\overline{\mathcal{A}}$  is a fundamental domain for the action of  $\widetilde{W}(R)$  on  $\mathfrak{h}_{\mathbb{R}}^*$

Each closed face (in the hyperplane arrangement  $\{H_{\alpha^\vee, m}\}$ ) is a  $\widetilde{W}(R)$  translate of a unique closed face of  $\overline{\mathcal{A}}$ .

In particular, FPP is tiled by translates of  $\overline{\mathcal{A}}$ .

**Corollary.** For every  $w$  in  $W(R)$ , there is a unique  $\tau(w) \in \mathbb{Z}\Xi$  with  $w\overline{\mathcal{A}} \subset \text{FPP}$ . These sets tile the FPP.

# Tiling the FPP



This is type  $B_2$  again. Red dots are the **root lattice**, spanned by  $(1, -1)$  and  $(0, 1)$ ; this is  $\mathbb{Z}^2 = \widetilde{W} \cdot (0, 0)$ .

White dots are complement of the root lattice in the weight lattice; this is  $\widetilde{W} \cdot (1/2, 1/2)$

$\mathcal{A}$  hatched **red**; alcoves in  $W \cdot \mathcal{A}$  hatched **black**, labels  $0 \dots 7$ .

Each FPP alc  $C$  shows **TWO** elts of  $W \cdot \mathcal{A}$  in  $C + \mathbb{Z}\Xi$ .

# One proven theorem to almost finish

Not my theorem. . . oh, well.

**Theorem.** Suppose  $G_{sc}$  is simply connected complex reductive. Then the fundamental parallelepiped (weights on which each simple coroot takes values in  $[0, 1]$ ) is the union of  $W(G_{sc})/Z(G_{sc})$  distinct translates of the fundamental alcove.

The theorem suggests that the character group  $Z(\widehat{G_{sc}})$  might naturally be a subgroup of the Weyl group. This is true, and the proof of the theorem shows how.

Here is some notation ( $G$  simple, simply connected).

1. Write  $S = \{\beta_1^\vee, \dots, \beta_\ell^\vee\}$  for the simple coroots, and  $\{\xi_i\}$  for the corresponding fundamental weights.
2. Write  $\beta_0^\vee$  for the lowest coroot, and  $\xi_0 = 0$ . Write

$$\widetilde{S} = \{\beta_0^\vee, \dots, \beta_\ell^\vee\}.$$

3. Define integers  $m_i$  ( $0 \leq i \leq \ell$ ) by
$$m_0 = 1, \quad \sum_{i=0}^{\ell} m_i \beta_i^\vee = 0.$$

# And a nice consequence actually to finish

This theorem was known when I was in preschool.

**Theorem.** Suppose  $G$  is simple complex reductive. Recall “extended coroots”

$$\widetilde{S} = \{\beta_0^\vee, \dots, \beta_\ell^\vee\}, \quad \sum_{i=0}^{\ell} m_i \beta_i^\vee = 0$$

Define  $W_Z = \{w \in W \mid w \cdot \widetilde{S} = \widetilde{S}\}$ .

1. Write  $\widetilde{S}_1 = \{i \mid m_i = 1\}$ . Then  $W_Z$  preserves  $\widetilde{S}_1$ , and its action there is faithful and simply transitive.
2. The stabilizer of the fundamental alcove  $\mathcal{A}$  in the extended affine Weyl group consists of the elements  $(w, -\xi_j)$ , with  $w \in W_Z$  and  $w \cdot \beta_0^\vee = \beta_j^\vee$ .
3. The weights  $\{\xi_i \mid i \in \widetilde{S}_1\}$  define all the characters of  $Z(G_{sc})$ , each exactly once.