

What does the unitary dual look like?

David Vogan, MIT

Recent developments in automorphic forms
and representation theory

AMS Eastern Sectional Meeting
SUNY Albany, October 20, 2024

Outline

What does the
unitary dual look
like?

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Introduction

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Your friend the Weyl group

Weyl group

My friend the affine Weyl group

Affine Weyl group

What do we know now about $\widehat{G(\mathbb{R})}_u$?

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Unitary dual II

The fundamental parallelepiped

FPP conjecture

The FPP conjecture

Slides at <http://www-math.mit.edu/~dav/paper.html>

What's this about **really**?

What does the
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$G(\mathbb{R})$ real reductive algebraic group.

$\widehat{G(\mathbb{R})}_u =$ (equiv classes of) **irr unitary reps of $G(\mathbb{R})$.**

I'll assume that studying this set (**unitary dual**) is the most world's best problem.

How can you approach it?

I'll start by answering the question in the title.

$G(\mathbb{R}) \rightsquigarrow \{\text{finite set of compact polyhedra } U_j\}.$

Each $U_j \rightsquigarrow$ (real vector space V_j , cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

Example of $SL(2, \mathbb{R})$

$G(\mathbb{R}) \rightsquigarrow \{\text{finite set of compact polyhedra } U_j\}.$

Each $U_j \rightsquigarrow (\text{real vector space } V_j, \text{ cone-in-a-lattice } C_j)$

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

$$SL(2, \mathbb{R}) \rightsquigarrow \left\{ \begin{array}{l} (\text{point}, \mathbb{R}^1, \{0\}) \longleftrightarrow \text{spherical unitary princ series} \\ (\text{point}, \mathbb{R}^1, \{0\}) \longleftrightarrow \text{nonsph unitary princ series} \\ (\text{point}, \mathbb{R}^0, \mathbb{N}) \longleftrightarrow \text{holomorphic discrete series} \\ (\text{point}, \mathbb{R}^0, \mathbb{N}) \longleftrightarrow \text{antihol discrete series} \\ ([0, 1], \mathbb{R}^0, \{0\}) \longleftrightarrow \text{complementary series} \end{array} \right\}$$

This is **two lines, two half lattices, and one interval**.

Picture for $SL(2, \mathbb{R})$ found by **Valentine Bargmann** in 1947.

For those with OCD or PhD: more words are needed to make this precise. Example: nonsph princ ser at 0 is **sum** of two irreps: $\text{nonsph}(\text{pt}, 0, 0) = \text{hol ds}(\text{pt}, 0, 0) + \text{antihol ds}(\text{pt}, 0, 0).$

That the picture works for **any** real reductive $G(\mathbb{R})$ comes from **Harish-Chandra, Langlands, Knapp, Zuckerman** about 1985.

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So what do we need to do?

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$G(\mathbb{R}) \rightsquigarrow \{\text{finite set of compact polyhedra } U_j\}.$

Each $U_j \rightsquigarrow (\text{real vector space } V_j, \text{ cone-in-a-lattice } C_j)$

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

Describe $\widehat{G(\mathbb{R})}_u \leftrightarrow$ describe cpt polyhedra U_j .

Vec space V_j , cone-in-lattice C_j important but easy.

Main question today: what do cpt polyhed U_j look like?

Answer: U_j is finite union of product of simplices.

Goals for today:

1. say what kinds of simplices are allowed
2. recall work of Barbasch, (Barbasch and his friends) giving beautiful precise list of simplices in many cases
3. say how `atlas` software computes ugly precise list of simplices in all cases

Remind me about the Weyl group...

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G cplx conn red alg group $\supset B$ Borel $\supset H$ max torus.

$(X^*$ alg chars of $H) \supset (R$ roots) $\supset (\Pi$ simple roots).

$(X_*$ alg cochars) $\supset (R^\vee$ coroots) $\supset (\Pi^\vee$ simple coroots).

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}_{\mathbb{R}}^* = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

$\mathfrak{h}_{\mathbb{R}}^*$ is the real vector space where the classical root system lives.

Root hyperplanes are $E_\alpha = \{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \gamma(\alpha^\vee) = 0\}$ (each α in R).

Each root α defines simple reflection: $\mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathfrak{h}_{\mathbb{R}}^*$,

$$s_\alpha(\gamma) = \gamma - \gamma(\alpha^\vee) \cdot \alpha, \quad s_\alpha(\alpha) = -\alpha, \quad s_\alpha = \text{identity on } E_\alpha.$$

Weyl group of G is $W =$ group generated by all s_α .

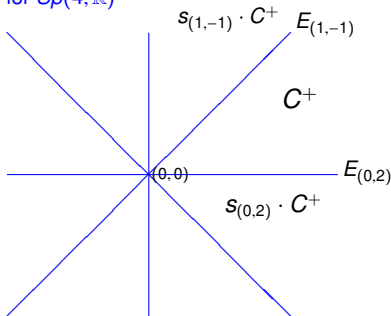
The open positive Weyl chamber is the open simplicial cone

$$C^+ = \{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \gamma(\alpha^\vee) > 0 \quad (\alpha \in \Pi \text{ simple})\}.$$

A Weyl chamber in $\mathfrak{h}_{\mathbb{R}}^*$ is a subset $w \cdot C^+$ (some $w \in W$).

What do Weyl chambers look like?

$\mathfrak{h}_{\mathbb{R}}^*$ for $Sp(4, \mathbb{R})$



$\overline{C}^+$ is fundamental domain for W action on $\mathfrak{h}_{\mathbb{R}}^*$.

Action of W on Weyl chambers is simply transitive

dominant faces of $\overline{C}^+$ of codim $d \longleftrightarrow$ cardinality d subsets of Π

any face of $\mathfrak{h}_{\mathbb{R}}^*$ is in $W \cdot (\text{unique dom face})$

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And the affine Weyl group?

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Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}_{\mathbb{R}}^* = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

Aff coroots are $R^{\vee, \text{aff}} = \{(\alpha^\vee, m) \mid \alpha^\vee \in R^\vee, m \in \mathbb{Z}\}$.

Pos aff coroots are $R^{\vee, \text{aff}, +} = \{(\alpha^\vee, m) \mid m > 0 \text{ or } \alpha^\vee \in R^{\vee, +}, m = 0\}$.

Write $\alpha_0^\vee = \text{lowest coroot}$ (unique since G simple).

Simple aff coroots are $\Pi^{\vee, \text{aff}} = \{(\alpha^\vee, 0) \mid \alpha^\vee \in \Pi^\vee\} \cup \{(\alpha_0^\vee, 1)\}$.

Aff hyperplanes $E_{\alpha, m} = \{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \gamma(\alpha^\vee) + m = 0\}$.

aff coroot $\rightsquigarrow$ simple aff reflection: $\mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathfrak{h}_{\mathbb{R}}^*$,

$$s_{\alpha^\vee, m}(\gamma) = \gamma - (\gamma(\alpha^\vee) + m) \cdot \alpha, \quad s_{\alpha^\vee, m} = \text{id on } E_{\alpha^\vee, m}.$$

Affine Weyl group of G is $W^{\text{aff}} = \text{group generated by all } s_{\alpha^\vee, m}$.

The open fundamental alcove is the open simplex

$$\begin{aligned} \mathcal{A}^+ &= \{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid \gamma(\alpha^\vee) + m > 0 \quad ((\alpha^\vee, m) \in \Pi^{\vee, \text{aff}} \text{ simple})\} \\ &= \{\gamma \in C^+ \mid \gamma(\alpha_0^\vee) < 1\}. \end{aligned}$$

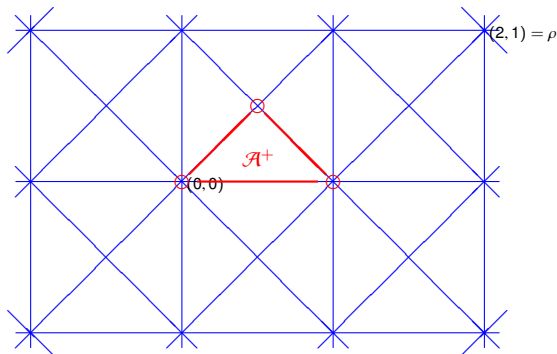
An alcove in $\mathfrak{h}_{\mathbb{R}}^*$ is a subset $w \cdot \mathcal{A}^+$ (some $w \in W^{\text{aff}}$).

What do alcoves look like?

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$\mathfrak{h}_{\mathbb{R}}^*$ for $Sp(4, \mathbb{R})$



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$\overline{\mathcal{A}}^+$ is **fundamental domain** for W action on $\mathfrak{h}_{\mathbb{R}}^*$.

Action of W^{aff} on alcoves is **simply transitive**

fund faces of $\overline{\mathcal{A}}^+$ of codim $d \longleftrightarrow$ order d subsets of $\Pi^{\vee, \text{aff}}$

any face of $\mathfrak{h}_{\mathbb{R}}^*$ is in $W^{\text{aff}} \cdot (\text{unique fundamental face})$

What good are all these faces?

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Langlands classif: irrs of real infl character indexed by

1. discrete parameter $(x, \lambda) \approx$ lowest K -type
2. continuous parameter $\gamma =$ infinitesimal character.

Here $x =$ KGB element: orbit of $K(\mathbb{C})$ on Borels in $G(\mathbb{C})$.

Finite # of x : 3 for $SL(2, R)$, 201 for $Sp(8, \mathbb{R})$, 320206 for split E_8 .

Given x , set of allowed λ is finite # of cones in lattices

Given (x, λ) , set of allowed γ is affine space $V_{\mathbb{R}}(x, \lambda) \subset \mathfrak{h}_{\mathbb{R}}^*$.

Therefore $V_{\mathbb{R}}(x, \lambda)$ is disjoint union of faces.

Theorem (Speh-V) Fix discrete parameter (x, λ) .

1. If $\gamma_1, \gamma_2 \in$ same face of $V_{\mathbb{R}}(x, \lambda)$, then irr reps $J(x, \lambda, \gamma_1)$ and $J(x, \lambda, \gamma_2)$ are both unitary or both nonunitary.
2. Set of unitary γ is a compact polyhedron
 $U(x, \lambda) \subset V_{\mathbb{R}}(x, \lambda)$, a finite union of faces.

What does that say about the unitary dual?

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Corollary Set $\widehat{G(\mathbb{R})}_{u,\text{real}}$ = unitary reps of real infl char. Then

$$\widehat{G(\mathbb{R})}_{u,\text{real}} = \bigcup_{x \in KGB} \bigcup_{\lambda \text{ allowed for } x} U(x, \lambda)$$

Claim in introduction:

$G(\mathbb{R}) \rightsquigarrow \{\text{finite set of compact polyhedra } U_j\}.$

Each $U_j \rightsquigarrow (\text{real vector space } V_j, \text{ cone-in-a-lattice } C_j)$

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

Polyhedra $U(x, \lambda)$ are the U_j in the introduction.

Extending **Cor** to **all** infl chars gives **real vector spaces** V_j .

Given x , λ 's are finite union of **cones in lattices** C_j .

To prove **Claim**, need to show **$U(x, \lambda)$ is nearly independent of λ** .

To describe unitary dual, need to **compute all** $U(x, \lambda)$.

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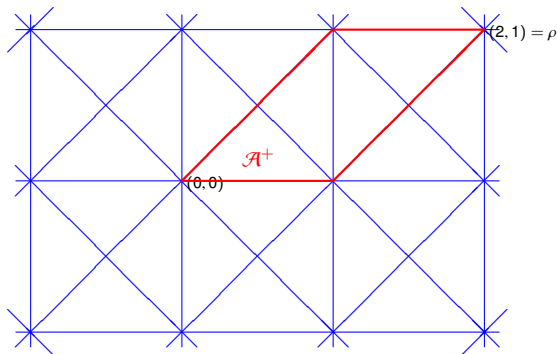
Unitary dual II

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FPP conjecture

What's the FPP...

FPP $\subset \mathfrak{h}_{\mathbb{R}}^*$ for $Sp(4, \mathbb{R})$



fundamental parallelepiped = $\{\gamma \in \mathfrak{h}_{\mathbb{R}}^* \mid 0 \leq \gamma(\alpha^\vee) \leq 1 \mid (\alpha \in \Pi)\}$

Union of $\#W/\#Z(G_{sc})$ alcoves.

$G(\mathbb{R})$	# alcoves	# faces
$SL(2, \mathbb{R})$	1	3
$Sp(4, \mathbb{R})$	4	19
split E_8	696729600	24169704765

(Those numbers are 7×10^8 and 2.4×10^{10} .)

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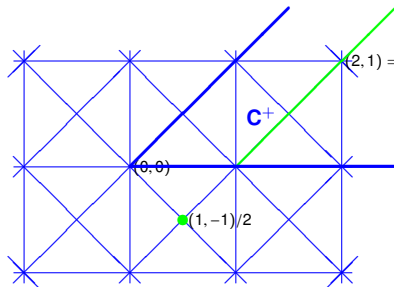
... and how does it help the unitary dual?

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Real Langlands parameter (x, λ, γ) defines

1. **Cartan involution** $\theta = \theta(x)$ acting on $\mathfrak{h}_{\mathbb{R}}^*$
2. **Cartan decomp** $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{t}_{\mathbb{R}}^* + \mathfrak{a}_{\mathbb{R}}^*$ (± 1 eigenspaces)
3. **differential of λ** $d\lambda \in \mathfrak{t}_{\mathbb{R}}^*$
4. **"A-parameter"** $\nu = \gamma(x, \lambda, \gamma) = \gamma - d\lambda$
5. Definition of param $\rightsquigarrow \gamma \in \overline{C^+}$ is **dominant**.



$(2, 1) = \rho$ (x, λ) first disc series, Siegel par

$$\theta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, d\lambda = (1/2, -1/2)$$

$$\mathfrak{a}_{\mathbb{R}}^* = \{(t, t)\}$$

green line is **allowed infl chars γ** .

unitary part is vertices $(2 + m_0, m_0)/2$,
edges $\{(1 + t, t) \mid t \in (1 + m_1/2, 1 + (m_1 + 1)/2)\}$,
some m_0 and m_1 in $\mathbb{N}$

Define $U_{FPP}(x, \lambda) = \{\gamma \in FPP \mid J(x, \lambda, \gamma) \text{ is unitary}\}.$

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The FPP conjecture

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Suppose (x, λ, γ) is a real Langlands parameter of infinitesimal character γ .

FPP conjecture is distilled from work of Dan Barbasch.

Define $S(\gamma) = \{\alpha \in \Pi \mid \gamma(\alpha^\vee) \leq 1\}$, a set of simple roots,

$\mathfrak{q} = \mathfrak{q}(\gamma) = \mathfrak{l} + \mathfrak{u}$ parabolic with Levi generated by $S(\gamma)$.

1. γ belongs to the FPP if and only if $\mathfrak{q} = \mathfrak{g}$.
2. Conjecture If $J(x, \lambda, \gamma)$ is unitary, then $\mathfrak{q}$ is θ -stable.
3. If $\mathfrak{q}$ is θ -stable, then $J(x, \lambda, \gamma)$ is good range cohomologically induced from $J(x_L, \lambda_L, \gamma_L)$ on L .
Here $\lambda_L = \lambda - \rho(\mathfrak{u})$, $\gamma_L = \gamma - \rho(\mathfrak{u})$, $\gamma_L \in FPP(L)$.

Assuming this conjecture,

$$U(x, \lambda) = \bigcup_{\theta\text{-stable } \mathfrak{q}} U_{FPP}(x_L, \lambda_L) + \rho(\mathfrak{u})$$

Conclusion: assuming conjecture, unitary dual is known if we compute (finitely many) $U_{FPP}(x, \lambda)$, the FPP infl characters for unitary reps in the series (x, λ) .