

Atlas of Lie Groups and Representations



www.liegroups.org

The Unitary Dual

Math Department, University of Maryland
September 17, 2025

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University of Maryland
Institute for Defense Analysis

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Jiu-Kang Yu

Gregg Zuckerman

Alessandra Pantano

Lie Groups

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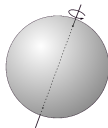
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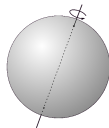
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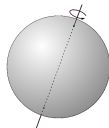


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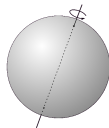
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Sophus Lie (1842-1899) was a Norwegian mathematician.

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Basic Principle:

we can only solve a hard math problem
if we can convert into linear algebra

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Atlas of Lie Groups and Representations

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Terminons en espérant que la classification du dual unitaire d'un groupe réductif quelconque sera bientôt achevée!

Outline

Atlas of Lie Groups and Representations (2002): use computers to study the unitary dual

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For several more mathematical presentations see www.liegroups.org/talks

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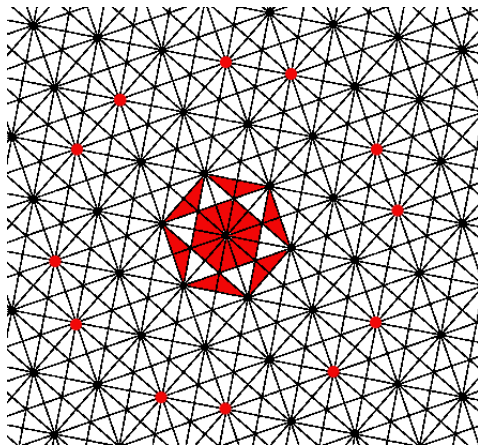
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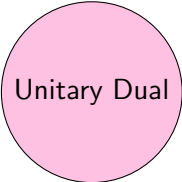
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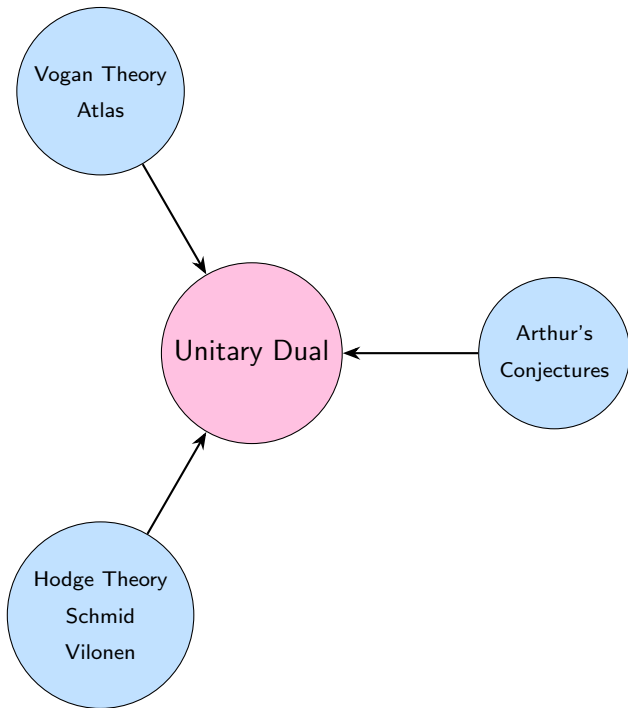
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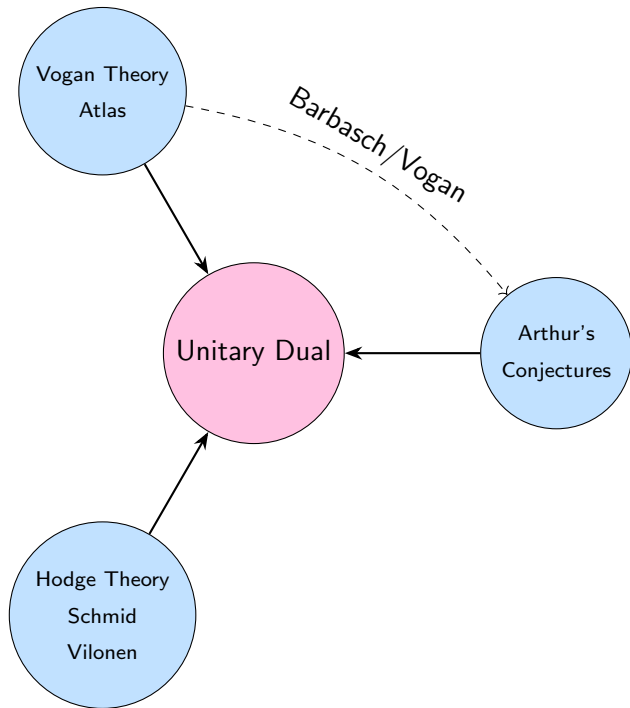
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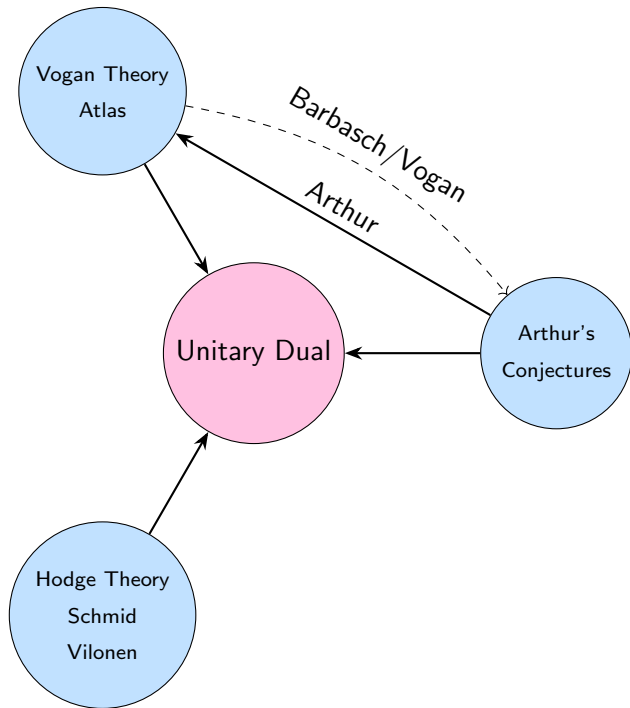


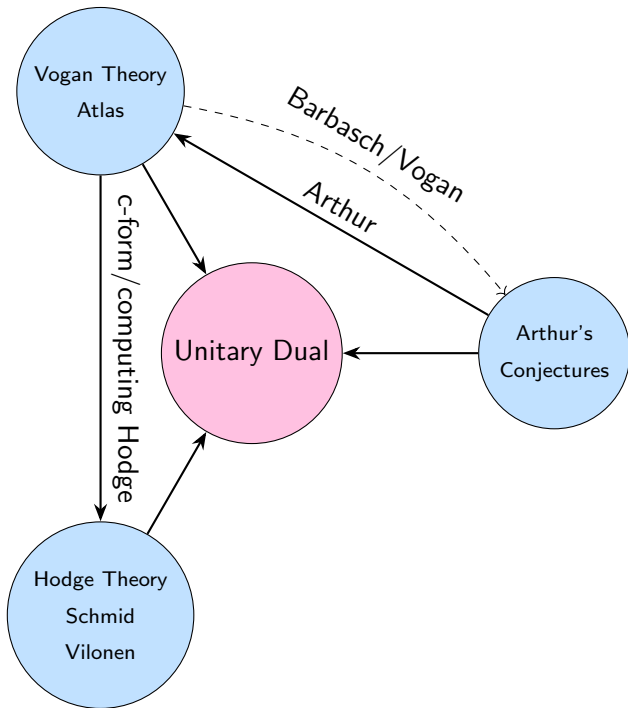


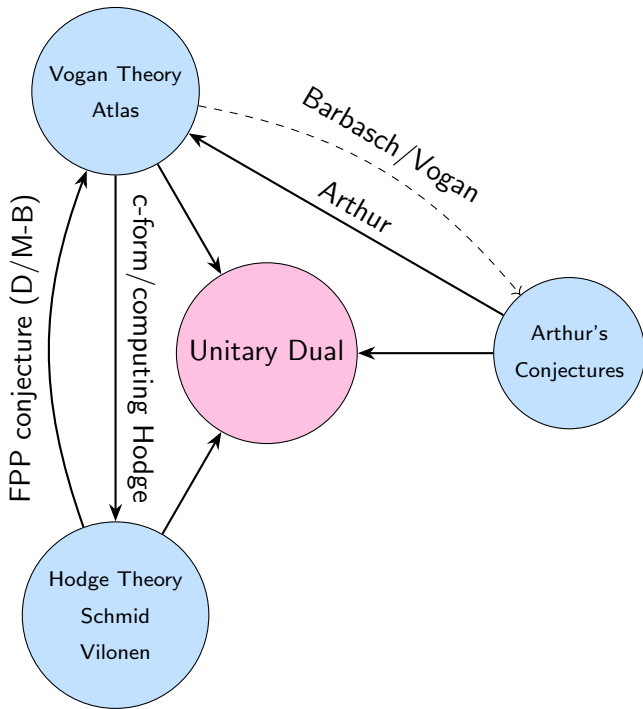
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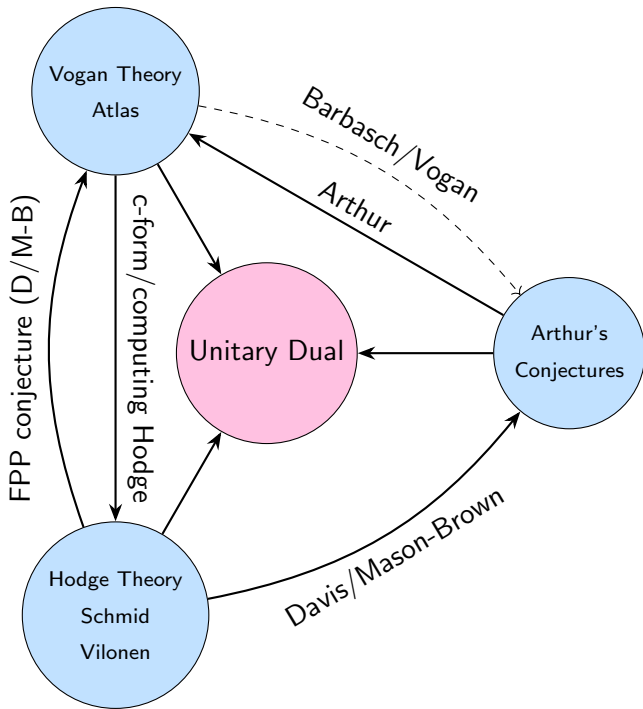












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The Langlands classification: finite parameter set $\mathcal{P}_\gamma$, and for each $\Gamma \in \mathcal{P}_\gamma$: $J(\Gamma)$ (irreducible) and $I(\Gamma)$ (standard)

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2007: *A Calculation the Size of Manhattan*: we computed the Kazhdan-Lusztig-Vogan polynomials for E_8 .





E_8 in the news

The E_8 story received wide coverage in newspapers and on radio and TV, as well as many on-line news sites. Examples include NPR's *All Things Considered*, ABC's *Good Morning America*, the Associated Press, and the New York Times. Congressman [Jerry McNerney](#) delivered a statement to Congress about the E_8 result. [Watch it on YouTube](#).

The preparations for the E_8 story were led by Brian Conrey and David Farmer from AIM, Jeff Adams from the University of Maryland, and Shari Boxer Baker from [JDS Group PR](#).

A sampling of news stories

The New York Times



Le Monde.fr



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$$\langle \pi(X)\vec{v}, \vec{w} \rangle + \langle \vec{v}, \pi(\sigma(X))\vec{w} \rangle = 0 \quad (\vec{v}, \vec{w} \in V, X \in \mathfrak{g})$$

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How do you compute the signature of a Hermitian form on an infinite dimensional vector space?

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- 2) The Hermitian form on $I(\Gamma)$ isn't canonical

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Remark: Around 2016 Wilfried Schmid told me: “I didn’t think this could be done.”

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2020: Based on some conjectures about the Hodge filtration, we formulated an algorithm to compute the Hodge filtration, and implemented this in **Atlas**.

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This is somewhat weaker than the original Schmid-Vilonen conjecture, but suffices for application to the unitary dual.

Describing the whole unitary dual

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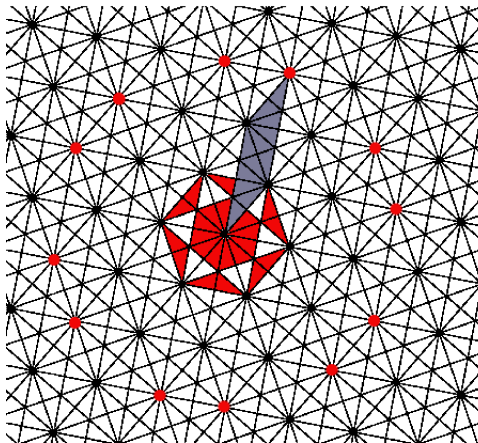
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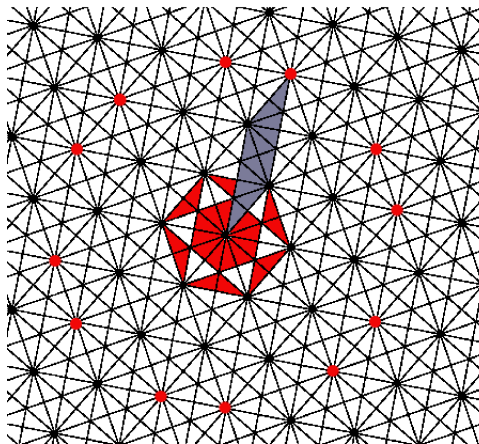
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Recall $\widehat{G}_u$ is the countable union of pictures like this:

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The **fundamental parallelepiped (FPP)** is the set in gray:

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FPP Conjecture (version 2): Induction from Levi subgroups induces:

$$\widehat{G}_u \leftrightarrow \bigcup_{L \subset G} \widehat{L}_{\text{FPP}}$$

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The proof is a beautiful application of Hodge theory.

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For each L , there is a finite set of hyperplane arrangements like the G_2 -picture to analyze.

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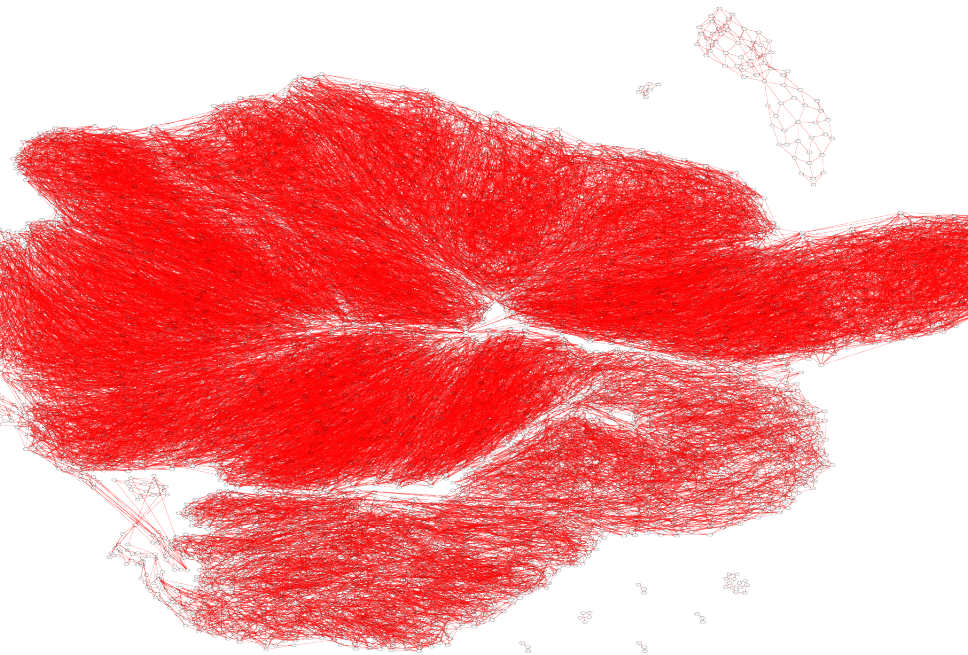
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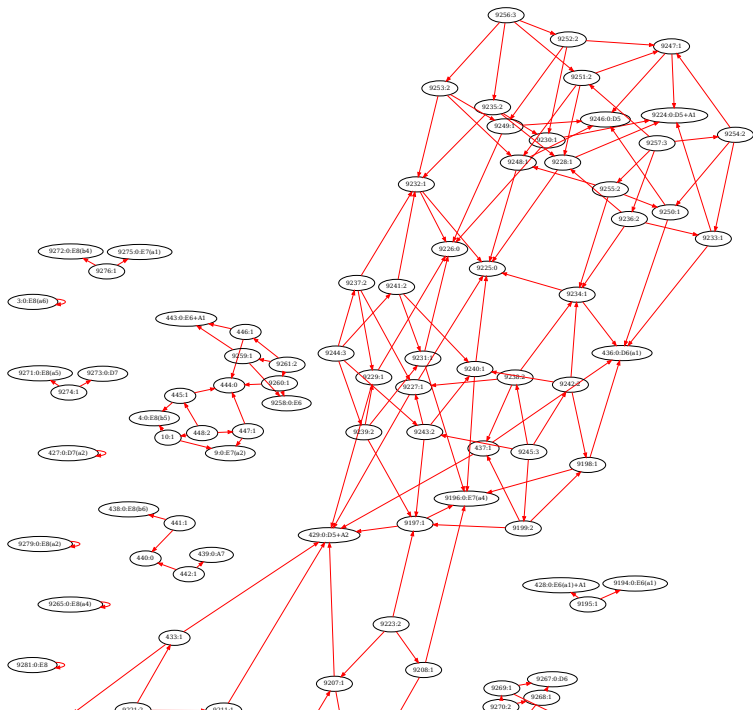
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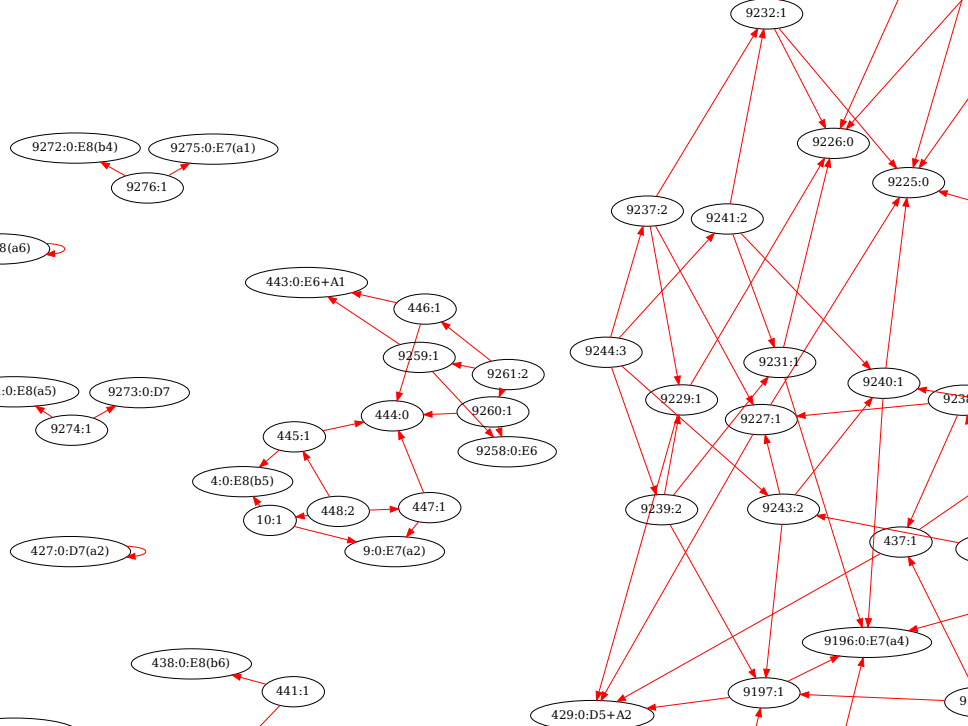
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Graph of the closure relations:







Description of the Unitary dual: Summary

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The FPP has a finite facet decomposition; unitarity is constant on facets; there is a finite calculation to compute $\widehat{L(\mathbb{R})}_{\text{FPP}}$ for each of the (finitely many) Q .

Some computer results

Groups up to rank 6 are quite fast on a laptop. . .

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$$E_6/E_7/E_8$$

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group	$\#(x, \lambda)$	#unitary	time (secs)
$E_6(split)$	26,325	2,217	98
$E_6(quat)$	74,459	19,831	662.316
$E_7(split)$	2,025,526	237,641	~ 16 hours/(1 hour parallel)
$E_8(split)$	~ 60 M	3,075,281	?

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Remark: Arthur representations account for most, but not all, of the unitary dual.



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Theorem:

- 1) **Unipotent case:** if Ψ is unipotent then $\Pi(\Psi)$ is unitary
- 2) **General Arthur packets:** In many cases $\Pi(\Psi)$ is known to be unitary

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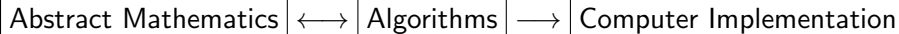
1)

Abstract Mathematics $\longleftrightarrow$ Algorithms

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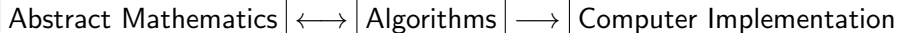
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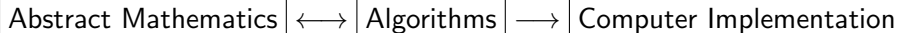


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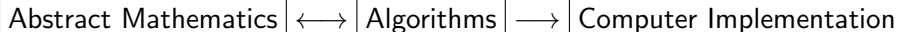


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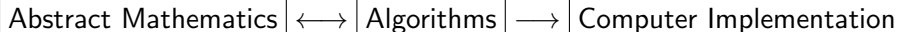


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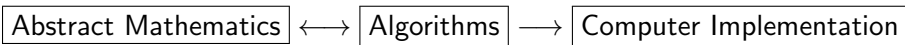


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- 2) Hard problems take a long time.

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b) You need a programmer who is also a mathematician.

2) Hard problems take a long time. Solving the unitary brought together deep results from representation theory, number theory, and Hodge theory, not to mention computer science

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a) Mainly the FPP for E_8 (i.e. confirming Steve Miller's answer)

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a) Possibly, but I'm not interested in doing this.

Thank you