

Atlas of Lie Groups and Representations



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Crux of the Biscuit¹: The Hard Parts of the Atlas Algorithm

Jeffrey Adams

University of Maryland
and
Institute for Defense Analysis

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¹Frank Zappa - Apostrophe (dedicated to Gregg Zuckerman)

The Atlas Algorithm

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This talk: the big ideas/key technical ingredients in the algorithm

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The **FPP unitary dual** is the set of irreducible unitary representations in the FPP. There is a finite calculation to compute this set (see David Vogan's talk).

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Together these results give a complete description of the unitary dual.

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This theorem has a complicated history and combines results of many authors: Arthur; Barbasch-Vogan; Barbasch; Adams-Arancibia-Mezo; Barbasch-Ma-Sun-Zhu; Adams-Miller-van Leeuwen-Vogan (atlas computation for exceptional groups); Davis-Mason-Brown

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Corollary: All Arthur packets for the real points of a connected reductive algebraic group are unitary.

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- Hodge theory and the Hodge filtration

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Changes in signature occur only at **reducibility points**

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This (sort of) gives an algorithm to compute the Hermitian form on standard modules (and eventually irreducibles. . .)

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Solution: the $\mathcal{C}$ -form.

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Defined so that: $\pi \simeq \pi^h$ if and only if π has an invariant Hermitian form.

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Corollary: If G is in the compact inner class then every representation (with real infinitesimal character) is Hermitian.

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This suggests looking at Hermitian forms collectively on real forms in a given inner class.

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So the signature has to do with the eigenvalues of $\pi(x)$.

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Theorem [Adams-Trapa-van Leeuwen-Vogan]:

(1) The c -form exists and is **unique** (up to positive scalar)

The c -form

The preceding discussion (and 1 million other ideas that were kicking around at the time) led to:

Definition: The c -form $(\ , \)_c$ on a $(\mathfrak{g}, K)$ -module satisfies:

$$(\pi(X)v, w)_c + (v, \pi(\sigma_c(X))w)_c = 0 \quad (X \in \mathfrak{g})$$

where σ_c is the compact form of G ,

normalized so $(\ , \)_c$ is **positive definite** on all lowest K -types of π

Theorem [Adams-Trapa-van Leeuwen-Vogan]:

- (1) The c -form exists and is **unique** (up to positive scalar)
- (2) The deformation algorithm gives a formula for the c -form on a standard representation in terms of c -forms on tempered representations.

Conclusion:

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$$\begin{aligned} \text{sig}^c(I_{1+\epsilon}) - \text{sig}^c(I_{1-\epsilon}) = \\ (1-s) \sum_{\substack{\phi,\tau \\ \ell(\gamma) - \ell(\tau) \text{ odd}}} P_{\phi,\tau}(s) Q_{\tau,\gamma}(s) \text{sig}^c(l(\phi)) \end{aligned}$$

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This is the key step in the algorithm for computing the c -form on a standard representation (and ultimately on irreducibles)

Who cares about the c -form?

What about the Hermitian form?

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If G is equal rank $\leftrightarrow \theta = \text{int}(x), x \in G$:

$$\langle \vec{v}, \vec{v} \rangle = \langle \vec{v}, \pi(x)\vec{v} \rangle$$

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Therefore: the c -form determines the Hermitian form in the equal rank case.

Serious problem: The c form does **not determine** the Hermitian form in the unequal rank case.

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In general: $\theta = \text{int}(y)$ for $y \in G\delta \subset G^\delta$. So: **we need to extend π to a representation of G^δ .**

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Which one? Recall (if π is Hermitian) it extends in 2 ways to G^δ : given by a choice of intertwining operator $T : \pi \simeq \pi^\delta$.

Extended Groups

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Which one? Recall (if π is Hermitian) it extends in 2 ways to G^δ : given by a choice of intertwining operator $T : \pi \simeq \pi^\delta$.

Bummer: There is no canonical way to choose T .

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Each Hermitian $\pi \mapsto \text{Ind}_G^{G^\delta}(\pi) = \{\pi_\pm\}$ where $\pi_- = \pi_+ \otimes \text{sgn}$.

Hope: although there is no canonical choice of $\pi_{\pm}$, choose $\pi_{0,+}$ for one representation π_0 in the block. Move it around by cross actions and Cayley transforms to define π_+ for all π .

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Consequently. . . ~ 2 years. . . *Parameters for Twisted Representations*
[Adams-Vogan]

Table 3 Cayley and cross actions on extended parameters: type 2

type	$\lambda _i = (\gamma - \rho) _i \quad (\vee \delta_0 - 1)\lambda = (1 + \vee \theta_y)\tau$ λ_1	τ_1	$\ell _a = (g - \rho^\vee) _a$ ℓ_1	$(\delta_0 - 1)\ell = (1 + \theta_x)t$ t_1	notes
2C crx	$w_\kappa \lambda + (\gamma_\alpha - 1)\kappa$	$w_\kappa \tau$	$w_\kappa \ell + (g_\alpha - 1)\kappa^\vee$	$w_\kappa t$	
2Ci Cay	$s_\alpha \lambda + (\gamma_\alpha - 1)\alpha$	$\tau - \frac{\tau_\alpha + \tau_\beta}{2} \alpha$	$s_\alpha \ell + (g_\alpha - 1)\alpha^\vee$	$s_\alpha t + (\ell_\alpha - g_\alpha + 1)\alpha^\vee$	$\ell_\alpha = \ell_\beta$
2Cr Cay	$s_\alpha \lambda + (\gamma_\alpha - 1)\alpha$	$s_\alpha \tau + (\lambda_\alpha - \gamma_\alpha + 1)\alpha$	$s_\alpha \ell + (g_\alpha - 1)\alpha^\vee$	$t - \frac{t_\alpha + t_\beta}{2} \alpha^\vee$	$\lambda_\alpha = \lambda_\beta$
2i11 crx	λ	τ	$\ell + \kappa^\vee$	t	
2i11 Cay	λ	$\tau - \tau_\alpha \sigma(\alpha) - \tau_\beta \sigma(\beta)$	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2} \alpha^\vee + \frac{g_\beta - \ell_\beta - 1}{2} \beta^\vee$	t	$\alpha = (1 + \vee \theta_{y_1})\sigma(\alpha)$
2i12 crlx	λ	τ	$\ell + \alpha^\vee$	$t - s$	$\alpha^\vee - \beta^\vee = s + \theta_x s$
2i12f Cay	$\lambda, \lambda + \alpha$	$\tau + \tau_\beta \sigma$ $-\frac{\tau_\alpha + \tau_\beta}{2} \alpha, \tau_1 - \sigma$	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2} \alpha^\vee$ $+ \frac{g_\beta - \ell_\beta - 1}{2} \beta^\vee$	t	$\tau_\alpha + \tau_\beta$ even $\alpha - \beta$ $= \sigma + \vee \theta_{y_1} \sigma$
2i12s Cay	$\lambda, \lambda + \alpha$	[none]	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2} \alpha^\vee + \frac{g_\beta - \ell_\beta - 1}{2} \beta^\vee$	t	$\tau_\alpha + \tau_\beta$ odd
2i22 Cay	$\lambda, \lambda + \kappa$ OR $\lambda + \alpha, \lambda + \beta$	$\tau - \frac{\tau_\alpha}{2} \alpha - \frac{\tau_\beta}{2} \beta$ OR $\tau - \frac{\tau_\alpha \pm 1}{2} \alpha - \frac{\tau_\beta \mp 1}{2} \beta$	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2} \alpha^\vee$ $+ \frac{g_\beta - \ell_\beta - 1}{2} \beta^\vee$	t	τ_α, τ_β even OR τ_α, τ_β odd
2r22 crx	$\lambda + \kappa$	τ	ℓ	t	
2r22 Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2} \alpha$ $+ \frac{\gamma_\beta - \lambda_\beta - 1}{2} \beta$	τ	ℓ	$t - t_\alpha s(\alpha^\vee)$ $- t_\beta s(\beta^\vee)$	$\alpha^\vee =$ $(1 + \theta_{x_1})s(\alpha^\vee)$
2r21 crlx	$\lambda + \alpha$	$\tau - \sigma$	ℓ	t	$\alpha - \beta = \sigma + \vee \theta_y \sigma$
2r21f Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2} \alpha$ $+ \frac{\gamma_\beta - \lambda_\beta - 1}{2} \beta$	τ	$\ell, \ell + \alpha^\vee$	$t + t_\beta s$ $- \frac{t_\alpha + t_\beta}{2} \alpha^\vee, t_1 - s$	$t_\alpha + t_\beta$ even $\alpha^\vee - \beta^\vee$ $= s + \theta_{x_1} s$
2r21s Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2} \alpha$ $+ \frac{\gamma_\beta - \lambda_\beta - 1}{2} \beta$	τ	$\ell, \ell + \alpha^\vee$	[none]	$t_\alpha + t_\beta$ odd
2r11 Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2} \alpha$ $+ \frac{\gamma_\beta - \lambda_\beta - 1}{2} \beta$	τ	$\ell, \ell + \kappa^\vee$ OR $\ell + \alpha^\vee, \ell + \beta^\vee$	$t - \frac{t_\alpha}{2} \alpha^\vee - \frac{t_\beta}{2} \beta^\vee$ OR $t - \frac{t_\alpha \pm 1}{2} \alpha^\vee - \frac{t_\beta \mp 1}{2} \beta^\vee$	t_α, t_β even OR t_α, t_β odd

Digression (a word from our sponsor)

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> I can't get the pdf `AdamsVoganChapter...pdf` to appear correctly.
I want it fill the page. Try fixing this

* Let me check the current state of that frame...

* The cleanest fix is a `[plain]` frame (no title/footer overhead)
with paper dimensions:...

> the image still only fills part of the screen. I think the pdf
image has large borders. I want to enlarge the image...

* Let me crop the PDF's whitespace automatically with `pdfcrop`,
then use the cropped version.

* `pdfcrop` automatically detected the tight bounding box of the
content and removed the surrounding whitespace...

> that's great, just add the header and footer back

Twisted KLV Polynomials

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Standard KLV polynomials $P_{\tau,\gamma} \in \mathbb{Z}[q]$ govern the Langlands classification:

$$J(\gamma) = \sum_{\tau} (-1)^{\ell(\gamma) - \ell(\tau)} P_{\tau,\gamma}(1) l(\tau)$$

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Conclusion: Redo the entire theory of KLV polynomials, and the algorithm, in the twisted setting.

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For tempered representations: the c -form is **positive definite**

Orientation Numbers

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Question: $\langle, \rangle_{\otimes}$ restricts to ± 1 the canonical form on $I(\lambda', \epsilon')$: determine the sign

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The matrix of the Casimir operator on this space is

$$C = H^2 + 2H + 4YX = \begin{pmatrix} \lambda^2 + 2j & 2(\lambda - j) \\ 2(\lambda + j) & \lambda^2 - 2j \end{pmatrix}$$

The eigenvectors and eigenvalues are:

$$\begin{aligned}\alpha_1 &= v_0 \otimes w_1 + v_2 \otimes w_{-1} && \text{eigenvalue } \lambda^2 + 2\lambda \\ \alpha_2 &= (\lambda - 1)v_0 \otimes w_1 + (\lambda + 1)v_2 \otimes w_{-1} && \text{eigenvalue } \lambda^2 - 2\lambda\end{aligned}$$

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and a short calculation shows

$$\begin{aligned}\langle \alpha_2, \alpha_2 \rangle &= 2\lambda(\lambda + 1)\langle v_2, v_2 \rangle \\ &= 2\lambda(\lambda + 1)\frac{\lambda - 1}{\lambda + 1}\end{aligned}$$

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Proposition: Consider $I(\lambda, 0)$, $0 < \lambda < n$

The sign relating the c-form on $I(\lambda, n+1)$ and the one coming from $I(\lambda, 0) \otimes F_n$ is

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This is a special case of the problematic **orientation numbers** which arise. They come from certain *non-integral* roots, and appear in various formulas as $(-1)^{\ell_0(\gamma) - \ell_0(\tau)}$

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This is a special case of the problematic **orientation numbers** which arise. They come from certain *non-integral* roots, and appear in various formulas as $(-1)^{\ell_0(\gamma) - \ell_0(\tau)}$

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Remark: The orientation numbers are what they are (essentially coming from the $SL(2, \mathbb{R})$ case above). There is no nice explanation of them (as far as I know).

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Deformation formula for the Hodge filtration:

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The c -form is the reduction mod 2 of the Hodge filtration

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It doesn't matter: we convert to the Hermitian form, which is definite on tempered representations.

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After a long digression through derived functors... this gives $\text{hodge}(\pi)$ in the tempered case.

Thank You